Phasors

Phasors and phasor math

A much more efficient way to analyze manipulations of digital audio

Phaser

Not a Phasor
Now, some simple questions?

What do I get if I add two sinusoids of the same frequency?

What do I get if I add two sinusoids that are different frequencies?
Sinusoidal math

How do we add sinusoids?

\[ a_1 \cos(\omega t + \phi_1) + a_2 \cos(\omega t + \phi_2) \]
\[ = a_1 \cos(\omega t) \cos(\phi_1) - a_1 \sin(\omega t) \sin(\phi_1) + a_2 \cos(\omega t) \cos(\phi_2) - a_2 \sin(\omega t) \sin(\phi_2) \]

And so on, quickly descending into a trigonometry hell even Dante could not conceive

There has got to be a better way!
Where does it all come from?

Unit circle: radius = 1
Rotation speed: $\omega$
(radians per second)
At the same time...

What if we decided to carry $\sin \omega t$ along as baggage?
Complex numbers

\[ j = \sqrt{-1} \]
\[ c = a + bj \] is a complex number

This is a two dimensional number system
Real axis –and- Imaginary axis

Niccolò Tartaglia
(1499–1577)
One of the first to use complex numbers
Imaginary numbers as 2d vectors

\[ z = x + jy \]

**Magnitude** is the vector length \( R = |z| = \sqrt{x^2 + y^2} \)

**Phase** is the angle relative to the real axis

We can also express in polar coordinates

\[ 2 + 4j \]

\[ -3 - 1j \]
Important facts about complex numbers

To add complex numbers
Add the two dimensions separately
Equivalent to vector addition

To multiply complex numbers
(x+jy)(v+jw)=(xv-yw)+j(xw+yv)
Equivalent to:
Multiplying the magnitudes,
Adding the phase
Sinusoids as complex numbers

\[ \cos \omega t + j \sin \omega t \]
Is there another expression?

Amazing facts dept...

\[
\frac{d}{d\theta} (\cos \theta + j \sin \theta) = -\sin \theta + j \cos \theta = j(\cos \theta + j \sin \theta)
\]

Derivative of a complex sinusoid

\[
\frac{d}{d\theta} e^{j\theta} = je^{j\theta}
\]

Derivative of something else
Euler’s Formula

Exponentials instead of \( \sin/\cos \) functions

\[
e^{j \theta} = \cos \theta + j \sin \theta
\]
Phasors

We’ve been talking about $\cos(\omega t)$ (well, $\sin(\omega t)$, same difference) as a sinusoid.

The equivalent (with the baggage) is:

$$e^{j\omega t}$$

This is a rotating vector

Called a Phasor
Phase shifts

\[ \cos(\omega t + \phi) \]
\[ e^{j(\omega t + \phi)} = e^{j \omega t} e^{j \phi} \]

To phase shift a phasor by \( \phi \), simply multiply by \( e^{j \phi} \)
Adding two sine waves

\[ a_1 \cos(\omega t + \phi_1) + a_2 \cos(\omega t + \phi_2) \]

\[
\begin{align*}
    &a_1 e^{j\omega t} e^{j\phi_1} + a_2 e^{j\omega t} e^{j\phi_2} \\
    &\left( a_1 e^{j\phi_1} + a_2 e^{j\phi_2} \right) e^{j\omega t} \\
    &\left( a_1 e^{j(\phi_1 - \phi_2)} + a_2 \right) e^{j\phi_2} e^{j\omega t}
\end{align*}
\]
What happens... 

\[ a_1 \cos(\omega t + \phi_1) + a_2 \cos(\omega t + \phi_2) \]

\[
\left( a_1 e^{j(\phi_1 - \phi_2)} + a_2 \right) e^{j\phi_2} e^{j\omega t}
\]
A really fun one...

\[ a_1 e^{j\omega t} + a_2 e^{j(\omega + \delta)t} \]

\[ = a_1 e^{j\omega t} + a_2 e^{j\omega t} e^{j\delta t} \]

\[ = \left( a_1 + a_2 e^{j\delta t} \right) e^{j\omega t} \]

What is this?
What happens...

\[ \left( a_1 + a_2 e^{j \delta t} \right) e^{j \omega t} \]

Magnitude

Radius = \( a_2 \)

Spinning at \( \delta \) radians per second

a_1
Beats

- The difference in frequency varies the amplitude
  - They are in phase, then out of phase, then...
  - The phenomena is called **beats**
Sampling a Phasor

We move $f/f_s$ of $2\pi$ for each sample.
Sampling near the Nyquist Frequency

\[ f_s/2 - \delta \]

\[ f_s/2 + \delta \]

\( \delta \) is some small amount
What about above $f_s$?

Rotation per sample is $\delta/f_s$.

Rotation per sample is $1 + \delta/f_s$. 
Frequency Folding

$0 \leq f \leq f_s/2$ : No problem

$f_s/2 < f \leq f_s$ : Aliases to $f_s-f$

$f_s < f \leq 3f_s/2$ : Aliases to $f-f_s$

And so on...

Negative frequencies fold back to positive

Note: Aliasing is fact, but not the enemy!
Delay

What if we add a delayed version of a signal to itself?

Flowgraph or flow diagram
Some conventions

- We’ll use $x$ for input and $y$ for output
  - $x_t$ is input at time $t$
  - $y_t$ is output at time $t$
- This is:
  - $y_t = x_t + a_1 x_{t-\tau}$
What does this do to a phasor?

- \( y_t = x_t + a_1 x_{t-\tau} \)
- \( y_t = e^{j\omega t} + a_1 e^{j\omega (t-\tau)} \)
  \[ = (1 + a_1 e^{-j\omega \tau}) e^{j\omega t} \]
  – Notice: Only phasor is dependent upon time

- What we care about most:
  – The “magnitude response”
  – Basically, what will happen to a given frequency!
Magnitude Response

- \( H(\omega) = 1 + a_1 e^{-j\omega\tau} \)
  - \( y_t = H(w)e^{j\omega t} \)

- Magnitude response is \( |H(\omega)| \)

- \( |H(\omega)| = (1 + a_1^2 + 2a_1 \cos(\omega\tau))^{1/2} \)

- Suppose \( a_1 = 1 \)
  - Response is zero at: \( \omega\tau = \pi, 3\pi, 5\pi, \) etc.
  - Response is 2 at: \( \omega\tau = 0, 2\pi, 4\pi, \) etc.
Plot example: $a_1 = 1$, $\omega t$ plotted
Plot example: $a_1 = -1, \omega t$ plotted

Example from the ToeTipper
Delay = 0.001, \( a_1 = 1 \), Freq plot
Inverse Comb Filters

- This type of filter is called an inverse comb filter, since it combs out harmonics.