Feedback Filters

• Why we might need feedback
• Uses for feedback filters
• Resons
Isotrack Motion Tracking System

• Tracks location of sensor relative to transmitter
  – X,Y,Z location
  – Pitch, Roll, Azimuth direction
• Up to 60 readings per second
• Tends to be a bit noisy
• How can we filter the noise?
Ways we know...

• Finite Impulse Response Filter
  – Filter that has a finite history of samples
  – What if we want one second of filtering?

\[ y_t = \sum_{i=0}^{N} a_i x_{t-i} \]
A new idea

• What if we used “feedback”?

\[ y_t = a_0 x_t + b_1 y_{t-1} \]

• This is a recursive structure.

\[ y_t = a_0 x_t + b_1 (a_0 x_{t-1} + b_1 (a_0 x_{t-2} + b_1 y_{t-3})) \]
One Example

\[ y_t = 0.5x_t + 0.5y_{t-1} \]
Infinite Impulse Response

• Filters with feedback have:
  – Infinite response to any input!
  – They store history in their state
Analyzing an IIR filter

• Suppose we have:

\[ y_t = x_t + b_1 y_{t-1} \]

• We can write as:

\[ y_t - b_1 y_{t-1} = x_t \]
\[ Y - b_1 z^{-1} Y = X \]
Solve for $Y$

\[ Y - b_1 z^{-1} Y = X \]
\[ Y (1 - b_1 z^{-1}) = X \]
\[ Y = \frac{1}{1 - b_1 z^{-1}} X \]
\[ \mathcal{H}(z) = \frac{1}{1 - b_1 z^{-1}} \]
Poles

\[
\mathcal{H}(z) = \frac{1}{1 - b_1 z^{-1}} = \frac{z}{z - b_1}
\]

\[\mathcal{H}(b_1) = \infty\]

We call these locations “poles”

Plot poles on the unit circle as an X

Suppose \(b_1 = 0.5\):

\[z = 0.5\]
Zeros Review

• Recall: we had zeros for non-feedback filters
  – \( y_t = x_t + 0.5x_{t-1} \)
  – \( \mathcal{H}(z) = 1 + 0.5z^{-1} \)
  – Zero is root of: \( z + 0.5 \), which is -0.5
  – Gain is the product of the distance to all zeros
Poles Contrast

- With feedback, we now have poles
  - $y_t = x_t + 0.5y_{t-1}$
  - $\mathcal{H}(z) = 1 / (1 - 0.5z^{-1})$
  - $\mathcal{H}(z)$ is infinite at: $z - 0.5$, which is 0.5
  - Gain is of the product of the inverses of the distances to all poles
Gain example

- Inverse of product of distance to poles
How to determine frequency response, revised

• Step 1: Determine transfer function
• Step 2: Get rid of negative exponents
• Step 3: Factor numerator, determine zeros
• Step 3b: Factor denominator, determine poles
• Step 4: Plot the pole and zeros on the z-plane
• Step 5: For every frequency, take the products of the distances to the zeros and inverse of distances to poles
Step 1: Determine the transfer equation

- $y_t = x_t + 0.5x_{t-1} + 0.5y_{t-1}$
  - $H(z) = (1+0.5z^{-1})/(1-0.5z^{-1})$
  - Rule: feedback terms in the denominators, **change the sign**
Examples

- \( y_t = x_t + 0.5x_{t-1} + 0.5y_{t-1} - 0.3y_{t-2} \)
  - \( \mathcal{H}(z) = (1 + 0.5z^{-1})/(1 - 0.5z^{-1} + 0.3z^{-2}) \)

- \( y_t = x_t - x_{t-1} + 0.4y_{t-2} \)
  - \( \mathcal{H}(z) = (1 - z^{-1})/(1 - 0.4z^{-2}) \)
Step 2: Get rid of negative exponents

- \( H(z) = \frac{1 + 0.5z^{-1}}{1 - 0.5z^{-1}} \) 
  
  \[ = \frac{z + 0.5}{z - 0.5} \]
Step 3: Factor numerator, determine zeros

- $\mathcal{H}(z) = \frac{z + 0.5}{z - 0.5}$
  - This is zero for $z = -0.5$, pole is 0.5
Step 4: Plot the poles and zeros on the z-plane

z = -0.5
z = 0.5
Step 5: Products of distances

- Frequencies are points on the circle
  - Distance to the points is the gain of the filter

\[
f = 0.2 \\
e^{0.2(2\pi)j} = \cos(0.2(2\pi)) + js\sin(0.2(2\pi))
\]
Did this filter have a pole?

- \( H(z) = \frac{z + 0.5}{z} \)
Stability

• What will this filter do?
  – \( y_t = x_t + 2y_{t-1} \)

• We refer to this filter as “unstable”
  – Commonly said to “blow up”
  – Too much feedback!!!
What’s that pole?

• $y_t = x_t + 2y_{t-1}$
  – What’s that pole?
Ensuring Stability

• A filter is stable if all poles are inside the unit circle!
  – Pole of 2 is not!
Contrasting poles and zeros

• Zeros decrease response as we get closer (down to zero)

• Poles increase response as we get closer (approaching infinity)
The Reson

• A useful filter
  – Place a pole at a certain frequency
  – Control how much that frequency is boosted

• A pole creates a “bump” in the frequency response
Bumps due to a pole
Where’s that there Pole?

\[ z = \text{Re}^{j\theta} \]

\[ H(z) = \frac{z}{(z - \text{Re}^{j\theta})} = \frac{1}{(1 - \text{Re}^{j\theta} z^{-1})} \]
Problem: This is complex

\[ H(z) = \frac{1}{(1 - \text{Re}^{j\theta} z^{-1})} \]

Solution: A symmetrical pole!
Multiply it out

\[ H(z) = \frac{1}{(1 - \text{Re}^{j\theta} z^{-1})(1 - \text{Re}^{-j\theta} z^{-1})} \]

\[ = \frac{1}{1 - 2R \cos \theta z^{-1} + R^2 z^{-2}} \]

\[ y_t = x_t + (2R \cos \theta) y_{t-1} - R^2 y_{t-2} \]
Designing a Reson Filter

• Choose a normalized frequency (0-0.5)
• Make it an angle \( \theta = 2\pi f \)
• Choose a radius: \( 0 < R < 1.0 \)
  – Closer to 1.0 is a sharper peak

• Step 4 had more implementation details.